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The field due to a pair of line vortices in a compressible fluid

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The problem considered is that of a self-propagating pair of line vortices, of equal and opposite strengths, in a compressible inviscid fluid. An asymptotic solution is sought in the limit of small propagation speed compared with the sound speed in the medium. In most of the fluid region, the elementary incompressible flow solution limit is enhanced by a Rayleigh–Janzen approximation. This approximation fails at points that are either far from, or very close to, a vortex. For distant points, rescaled outer variables lead to an approximation that corresponds to the field induced by a moving dipole in compressible fluid. The main approximation also fails near each vortex line, where the analysis of Barsony-Nagy, Er-El & Yungster (*J. Fluid Mech.* vol. 178, 1987, p. 367) is used to express the local flow field in terms of a hypergeometric function. A particular feature of the problem is the propagation parameter P, which is proportional to $U'h'/K$, where U' and $2h'$ denote the propagation speed and separation of the vortices and K is the circulation. The parameter P is a function of the Mach number M and has the asymptotic value unity in the limit of incompressible flow. The analysis leads to the conclusion that $P = 1 + o(M^2)$ for small values of M; that is, the propagation number is unchanged to order $O(M^2)$. This differs from earlier work, which predicted the asymptotic development $P \sim 1 - M^2/4$.

1. Introduction

The study of vortex dynamics has played a central role in the analysis of fluid mechanics, and there are many examples where its investigation can provide significant insights into the structure of fluid flow fields.

For example, the sound field induced by a fluid motion has been shown to be equivalent to that produced by a hypothetical source distributed related to the divergence of the vector product of vorticity and velocity fields (Powell 1964; Howe 1975). In some circumstances, this gives a convenient alternative to the renowned formulation of Lighthill (1952) who expresses the sound field in terms of a hypothetical distribution of quadrupole sources related to the local velocity field.

Representations of fluid motions in terms of the vorticity field are particularly useful if the vorticity is confined to relatively small regions. A classical idealized model is that of a vortex filament, with the region of vorticity taken to be a line of zero thickness. This has the advantage that the corresponding velocity field can be calculated relatively easily, at least in the case of incompressible inviscid flow, in the form of a line integral using the Biot-Savart formula (Saffman 1993; Kambe 2004). The motion of line vortices in isentropic flow, with given solid boundaries, can be analysed using the notion that vortex lines move with the fluid, with care taken to allow appropriately for the inevitable singularity at the vortex itself, and much progress has been made in methods to track the vortex paths (see Saffman 1993 for example).

This simple model of line vortices in incompressible flow has the considerable advantage of being amenable to the detailed calculations described above; but the model fails at points that are too far from, or too close to, the vortices. The incompressible flow approximation breaks down at sufficiently great distances (from the vortices) where sound waves generally will be present. The elementary solution also breaks down at points close to a line vortex. Work due to Taylor (1930) provides the form of the governing equation for points at distances of order $O(M)$ from a straight line vortex.

The line vortex model is inadequate as the distance to a vortex becomes vanishingly small and we must replace the elementary model with one in which there is a core of small but finite radius, such as a core of stagnant fluid (see Moore & Pullin 1987 for example) or a hollow core of small diameter (Leppington & Sisson 1997; Pocklington 1894) or else a light cylinder that drifts along with the fluid. A significant complication associated with the 'fluid-core' model is that its shape is an unknown of the problem. Moore & Pullin (1987) address the problem of a pair of self-propagating vortex cores containing stagnant constant-pressure fluid (and with equal and opposite strengths); they present numerical solutions for a range of values of two independent parameters of the problem. Their work also includes an analysis of the limiting case where the (maximum) diameter of the vortices is small.

The present work returns to the simplest non-trivial problem including compressibility: a pair of straight line vortices, with equal and opposite (circulation) strengths $\pm K$, propagates in an inviscid fluid in a direction perpendicular to the line joining the vortices.

The significant physical quantities in this steady compressible-flow problem are the mean sound speed c_0, the circulation constant K, the separation $2h'$ and the propagation speed U' of the vortex pair. These form two dimensionless parameters, namely a 'propagation number'

$$P = 4\pi U'h'/K \tag{1.1}$$

and a Mach number defined as

$$M = U_0/c_0, \tag{1.2}$$

where U_0 is the value of the propagation speed in the limit of incompressible flow. The two dimensionless numbers must be related: thus $P = P(M)$, where the function $P(M)$ has to be determined as part of the problem. In the the limit of incompressible flow, the problem is elementary, with $U' = K/(4\pi h')$, hence $P \sim 1$ as $M \to 0$.

The possible variation of P due to small compressibility effects is noted by Barsony-Nagy, Er-El & Yungster (1987), and by Moore & Pullin (1987). The latter authors investigate the problem of the steady self-propagation of a symmetrically shaped pair of vortices in compressible fluid. They derive numerical results for several vortex core sizes and Mach numbers and include an analysis for the limiting case of vortex filaments, with the prediction that $P \sim 1 - M^2/4$ as $M \to 0$. The comparison with their numerical results is not conclusive because of numerical difficulties associated with the singular behaviour of the line vortex model. The different conclusion reached in the present analysis is that there is no perturbation in the propagation parameter P to the order of approximation ($O(M^2)$) considered here. That is, it is predicted here that

$$P = 1 + o(M^2) \quad \text{as} \quad M \to 0. \tag{1.3}$$

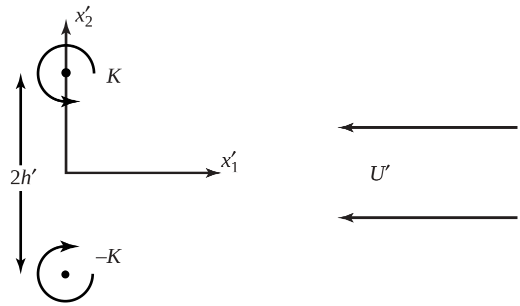

FIGURE 1. The geometry of the problem, shown in a reference frame that is fixed relative to the vortices.

An asymptotic analysis is given for the limit of small Mach number M, the procedure being as follows. In most of the fluid region (called the 'main' region) the elementary textbook solution for incompressible flow is the leading-order approximation, and this is enhanced in the analysis of § 2 by a 'Rayleigh–Janzen' correction term proportional to M^2, where M is the Mach number. This approximation fails at points that are either far from, or very close to, a vortex. For distant points, rescaled variables are defined in § 3, leading to an outer approximation that can be interpreted as being the potential due to a steadily moving dipole in compressible fluid. A physical argument is used to reproduce the dipole approximation obtained by the formal matching procedure described above.

Barsony-Nagy *et al.* (1987) point out that the incompressible-flow approximation also fails near a vortex line and they use Taylor's equation (Taylor 1930) to express the local flow field in terms of a hypergeometric function. Their method is described in § 4 and is used to express the form of the solution near each vortex.

2. Main approximation

Cartesian coordinates (x_1', x_2') are chosen so that the plane of symmetry is given by $x_2' = 0$ and the vortices are at $(U't', \pm h')$ at time t'. Equivalently, in a reference frame that moves with the vortices, the vortices are at rest and the distant fluid has velocity $(-U', 0, 0)$ (see figure 1). In view of the symmetry, attention can be confined to the upper vortex at $(U't', h')$. The circulation round this vortex has the prescribed value K, which is expressed for later convenience as $K = 2\pi\kappa$. The propagation speed U' of the vortices in the positive x_1'-direction is constant throughout the motion.

In the limit of incompressible flow, the upper vortex has height given by $h' \sim h_0$ and is driven by the image vortex at $y = -h' \sim -h_0$, with circulation $-2\pi\kappa$ and with h_0 related to the propagation speed $U' \sim U_0$ by the expression

$$U_0 = \frac{\kappa}{2h_0}. \tag{2.1}$$

A dimensionless Mach number M is defined as

$$M = \frac{U_0}{c_0} = \frac{\kappa}{2h_0 c_0}, \tag{2.2}$$

where c_0 denotes the sound speed of infinitesimal vibrations. The separation distance $2h'$ and propagation speed U' can differ from their limiting values $2h_0$ and U_0 when compressibility effects are taken into account.

It was remarked in §1 that the parameters κ, h' and U' form a dimensionless number

$$P = \frac{2U'h'}{\kappa}, \tag{2.3}$$

which is a function of the Mach number M. Thus, $P = P(M)$ with the limiting value

$$P \sim 1 \quad \text{as} \quad M \to 0, \tag{2.4}$$

and the possible variation of P from this limiting value is considered later.

The circulation constant κ is taken to be fixed. In addition, one of the parameters h' and U' can be specified and the other has to be determined as part of the solution. For a given value of the Mach number M, the product $U'h'$ is an invariant, according to (2.3). We could take either $h' = h_0$ or $U' = U_0$ at the outset, with the other one to be determined, but it is instructive to allow both h' and U' to differ slightly from their limiting values h_0 and U_0.

The general unsteady equation satisfied by the velocity potential ϕ', for isentropic gas, is that given by Blokhintsev (1946), (see also Longhorn 1952; Lighthill 1972), namely

$$\left\{c_0^2 - (\gamma-1)\left(\frac{\partial\phi'}{\partial t'} + \frac{1}{2}\frac{\partial\phi'}{\partial x_j'}\frac{\partial\phi'}{\partial x_j'}\right)\right\}\frac{\partial^2\phi'}{\partial x_i'^2} = \frac{\partial^2\phi'}{\partial t'^2} + 2\frac{\partial\phi'}{\partial x_i'}\frac{\partial^2\phi'}{\partial x_i'\partial t'} + \frac{\partial\phi'}{\partial x_i'}\frac{\partial\phi'}{\partial x_j'}\frac{\partial^2\phi'}{\partial x_i'\partial x_j'}, \tag{2.5}$$

where the double suffix summation convention is used, with indices i and j running from 1 to 2. The term inside the curly brackets, on the left-hand side of the equation, is the variable sound speed; γ is the ratio of specific heats; $c_0^2 = \mathrm{d}p/\mathrm{d}\rho$, evaluated at the equilibrium values of pressure p and density ρ, is the square of the sound speed for infinitesimal vibrations.

The same equation holds for either of two possible reference frames, which are fixed with respect to either the moving vortices or the fluid at infinity; the potential refers to the velocity field relative to the chosen reference frame. In the steady problem considered here, it is convenient to work in the frame fixed relative to the vortex pair, so that the potential is independent of time, and three of the terms in equation (2.5) disappear.

Given the length scale h_0 and circulation $2\pi\kappa$, a time scale h_0^2/κ can be formed and dimensionless variables Φ, X_i, t, h and U are defined by the formulae

$$\phi' = \kappa\Phi, \quad x_i' = h_0 X_i, \quad t' = \left(h_0^2/\kappa\right)t, \quad h' = h_0 h, \quad U' = 2U_0 U. \tag{2.6}$$

In terms of these variables, equation (2.5) takes the form

$$\nabla^2\Phi = 4M^2\left\{\frac{\partial\Phi}{\partial X_i}\frac{\partial\Phi}{\partial X_j}\frac{\partial^2\Phi}{\partial X_i\partial X_j} + \tfrac{1}{2}(\gamma-1)\nabla^2\Phi\left(\frac{\partial\Phi}{\partial X_j}\frac{\partial\Phi}{\partial X_j}\right)\right\}, \tag{2.7}$$

where Φ is the potential of the velocity field relative to the fixed vortex pair. Thus, at a great distance from the vortices, we have $\phi' \sim -U'x_1'$, hence

$$\Phi \sim -\frac{U'}{2U_0}X_1 \equiv -UX_1 \quad \text{as} \quad R \equiv \left(X_1^2 + X_2^2\right)^{1/2} \to \infty. \tag{2.8}$$

with U defined as $U'/2U_0$. In most of the flow domain (namely at points not too close to, or too far from, the vortex), the solution is given asymptotically, for small values of the Mach number M, by an elementary incompressible flow potential. Thus, we pose the Rayleigh–Janzen expansion

$$\Phi \sim \Phi_0 + M^2\Phi_1 + \cdots, \tag{2.9}$$

where Φ_0 satisfies the Laplace equation, with

$$\Phi_0 \sim -\tfrac{1}{2}X_1 \quad \text{as} \quad R \sim \infty, \tag{2.10}$$

and with appropriate singularities near the vortex and its image. This elementary incompressible-flow problem has the well-known solution

$$\Phi_0 = \ \mathrm{Re} F(Z) \equiv \tfrac{1}{2}\{F(Z) + \overline{F}(\overline{Z})\}, \tag{2.11}$$

where $Z = X_1 + \mathrm{i}X_2$ and

$$F(z) = -\mathrm{i}\log(Z - \mathrm{i}) + \mathrm{i}\log(Z + \mathrm{i}) - \tfrac{1}{2}Z + \tfrac{1}{2}\pi, \tag{2.12}$$

and the constant term has been added for later convenience. A correction to the approximation $\Phi \sim \Phi_0$ is obtained from the Rayleigh–Janzen expansion (2.9), and substitution into (2.7) leads to the following equation for the correction potential Φ_1:

$$\nabla^2\Phi_1 = 2\nabla\Phi_0 \cdot \nabla((\nabla\Phi_0)^2), \tag{2.13}$$

with $|\nabla\Phi_1| = O(1)$ (to ensure a uniform steady flow) as $R \equiv (X^2 + Y^2)^{1/2} \to \infty$.

Equation (2.13) is simplified by changing from the independent variables X_1 and X_2 to $Z = X_1 + \mathrm{i}X_2$ and $\overline{Z} = X_1 - \mathrm{i}X_2$ (see Barsony-Nagy *et al.* 1987). We find that (2.13) takes the form

$$\nabla^2\Phi_1 = 2F'(Z)^2\overline{F}''(\overline{Z}) + 2\overline{F}'(\overline{Z})^2F''(Z). \tag{2.14}$$

Thus, we have

$$\Phi_1 = \Omega_1(Z, \overline{Z}) + \Omega_2(\overline{Z}, Z), \tag{2.15}$$

where

$$2\frac{\partial^2\Omega_1}{\partial Z\partial\overline{Z}} = F'^2\overline{F}'', \quad 2\frac{\partial^2\Omega_2}{\partial Z\partial\overline{Z}} = \overline{F}'^2F''. \tag{2.16}$$

The function Ω_2 is the complex conjugate of Ω_1.

A particular solution Φ_p for Φ is calculated first, noting that the general solution to (2.13) is then obtained by adding a harmonic function. Direct integration of (2.16) (see Barsony-Nagy *et al.* 1987) shows that

$$2\Omega_{1\mathrm{p}} = \overline{F}'(\overline{Z}) \int F'(Z)^2\,\mathrm{d}Z. \tag{2.17}$$

With $F(Z)$ given explicitly by (2.12), the integration is elementary and a particular solution Φ_p follows immediately. It is convenient to add a harmonic function $(Z + \overline{Z})/16$, to obtain a particular solution that vanishes as $|Z| \to \infty$. That is,

$$4\Phi_\mathrm{p} = \frac{Z^3 - 2Z\overline{Z}^2 + 7Z + \overline{Z}^3 - 2\overline{Z}Z^2 + 7\overline{Z}}{(Z^2+1)(\overline{Z}^2+1)}, \tag{2.18}$$

and this is the solution obtained by Moore & Pullin (1987). It is emphasized, however, that (2.18) is not the only solution of (2.13) – it is not even the only solution that vanishes at infinity – as eigensolutions (harmonic functions) can be added.

Permissible eigensolutions must have even symmetry about the plane $X_2 = 0$ (so that $\partial\Phi_1/\partial X_2 = 0$ there). Furthermore, for consistency with subsequent matching requirements, they can be no larger than $O(Z)$ at infinity and no worse than $O((Z\mp i)^{-1})$ as $Z \to \pm i$. Thus, the permissible eigensolutions are the functions

$$\Phi_\mathrm{A} = \frac{Z}{Z^2+1} + \frac{\overline{Z}}{\overline{Z}^2+1} \tag{2.19}$$

and

$$\Phi_{\rm B} = Z + \overline{Z}. \tag{2.20}$$

The proposed form for the Rayleigh–Janzen correction term Φ_1, of (2.9), is

$$\Phi_1 = \Phi_{\rm p} + A\,\Phi_{\rm A} + B\,\Phi_{\rm B}, \tag{2.21}$$

where the real constants A and B have to be found.

The singular behaviour of $\Phi_{\rm A}$ at $Z = {\rm i}$ (and at the image point $Z = -{\rm i}$) corresponds to dipole behaviour there. The particular solution (2.18) also has such dipole singularities, so the added term $A\,\Phi_{\rm A}$ changes the scale, but not the nature, of this singularity. Moore & Pullin (1987) noted that such a dipole singularity can be removed by a slight change of position of the vortex near $Z = i$, placing it at the nearby point $(\hat{X}_1, \hat{X}_2) = (X_1, X_2 - \delta M^2)$, with δ chosen suitably, and this corresponds to a shift in the distance from the vortex to the symmetry plane, leading to the reduced value $h' \sim h_0(1 - \delta M^2)$ due to the effect of compressibility. They inferred the value $\delta = 1/4$.

Here, we consider the more general form (2.21) for the correction potential Φ_1 and point out that either of the real constants A or B can be chosen arbitrarily, with the other constant to be determined. The reason for this flexibility is a direct consequence of the original problem having a free parameter: with circulation fixed, we may specify either the speed of the vortex or else its distance h' from the plane. If we take h' to have the fixed value h_0, then the compressibility correction leaves the vortex position unchanged and this amounts to choosing the constant A in (2.21) to remove the singularity at $Z = \pm i$: it transpires that we would require $A = -1/4$ in that case. Then compressibility effects might change the vortex propagation speed from its limiting value U_0 given by (2.1) in the incompressible limit and any such perturbation is related to the value of the other constant B in (2.21). Alternatively, we could insist that the propagation speed has the fixed value $U' = U_0$ (that is, $U = 1/2$), and subsequent calculations are then required to determine any change in distance of vortex from the plane, in the manner described above.

In the present analysis, neither A or B will be fixed at the outset, so both h' and U' will contain a free parameter, though the dimensional propagation number P (see (2.3)) will not depend on that parameter.

2.1. *Behaviour near the upper vortex*

Near the upper vortex (at $X_1 = 0$, $X_2 = 1$ in our main coordinate system), we use local polar coordinates (R_1, θ_1), with $R_1 = (X_1^2 + (X_2 - 1)^2)^{1/2}$ and $\theta_1 = \arctan((X_2 - 1)/X_1)$ and expand expression (2.9) as $R_1 \to 0$, to obtain

$$\Phi \sim \theta_1 - \tfrac{1}{8}R_1^2 \sin 2\theta_1 + M^2\left\{ \left(A + \tfrac{1}{4}\right)\frac{\cos\theta_1}{R_1} - \tfrac{3}{4}\sin 2\theta_1 + \left(\tfrac{1}{4}A + 2B + \tfrac{1}{16}\right) R_1\cos\theta_1 - \tfrac{1}{4}R_1\cos 3\theta_1 \right\}. \tag{2.22}$$

The singular term proportional to $R_1^{-1}\cos\theta_1$ arises because the vortex position, at $X_2 = 1$ in the incompressible limit, is modified at order $O(M^2)$ owing to compressibility effects. To eliminate this singular term, we follow Moore & Pullin (1987) by shifting to a displaced vortex location at $X_2 = 1 - M^2\delta$, say. Thus, displaced polar coordinates are specified by the formulae $R_1\cos\theta_1 = \hat{R}\cos\hat{\theta}$, $R_1\sin\theta_1 = \hat{R}\sin\hat{\theta} - \delta$.

The approximation (2.22) can readily be rewritten in terms of the displaced polar coordinates $(\hat{R}, \hat{\theta})$ and the singular term is eliminated by the choice

$$\delta = A + \tfrac{1}{4}. \tag{2.23}$$

Thus, we find that the solution near the upper vortex has the form

$$\Phi \sim \hat{\theta} - \tfrac{1}{8}\hat{R}^2 \sin 2\hat{\theta} + M^2\left\{-\tfrac{3}{4}\sin 2\hat{\theta} + \left(\tfrac{1}{2}A + 2B + \tfrac{1}{8}\right)\hat{R}\cos\hat{\theta} - \tfrac{1}{4}\hat{R}\cos 3\hat{\theta}\right\}. \quad (2.24)$$

The term proportional to $M^2\hat{R}\cos\hat{\theta} = M^2 X_1$ corresponds to a steady flow at the location of the upper vortex. Noting that vortex lines move with the fluid (or equivalently, the net force is zero on a small cylinder centred at the vortex), this term must vanish and we have

$$B = -\tfrac{1}{4}A - \tfrac{1}{16}, \quad (2.25)$$

leading to

$$\Phi \sim \hat{\theta} - \tfrac{1}{8}\hat{R}^2 \sin 2\hat{\theta} + M^2\left\{-\tfrac{3}{4}\sin 2\hat{\theta} - \tfrac{1}{4}\hat{R}\cos 3\hat{\theta}\right\}, \quad (2.26)$$

near the upper vortex.

The displacement factor δ of (2.23) also implies that the upper vortex has the vertical location given by

$$h'/h_0 \equiv h = (1 - \delta M^2), \quad (2.27)$$

to order M^2, with $\delta = A + 1/4$ according to (2.23).

2.2. *Limit as $R \to \infty$*

The main approximation to order $O(M^2)$ has the form (2.9) with Φ_0 and Φ_1 given by (2.11) and (2.21). At large distances, where $R \equiv (X_1^2 + X_2^2)^{1/2} \to \infty$, and $\theta = \arctan(X_2/X_1)$ is the usual polar coordinate angle, we find

$$\Phi \sim -\tfrac{1}{2}R\cos\theta - 2\frac{\cos\theta}{R} + M^2\left\{-\left(\tfrac{1}{2}A + \tfrac{1}{8}\right)R\cos\theta + (2A-1)\frac{\cos\theta}{R} + \frac{1}{2}\frac{\cos 3\theta}{R}\right\}, \quad (2.28)$$

where use has been made of the relationship (2.25) for B in terms of A. Expression (2.28) gives an estimate for the velocity potential of the fluid motion relative to the centre of the vortex pair. The potential relative to the fluid at infinity is given to this order by subtracting the terms proportional to $X_1 = R\cos\theta$. Thus,

$$\widehat{\Phi} = \Phi + \left\{\tfrac{1}{2} + \left(\tfrac{1}{2}A + \tfrac{1}{8}\right)M^2\right\}R\cos\theta, \quad (2.29)$$

and the flow at infinity (relative to the vortices) has the magnitude

$$\frac{U'}{2U_0} \equiv U \sim \tfrac{1}{2} + \left(\tfrac{1}{2}A + \tfrac{1}{8}\right)M^2, \quad (2.30)$$

in place of the value $U \sim 1/2$ in the incompressible flow limit.

Expressions (2.27) and (2.29) combine to show that the propagation parameter P (defined by (1.1)) has the development

$$P = 1 + o(M^2) \quad \text{as} \quad M \to 0, \quad (2.31)$$

which is, of course, independent of the choice of the constant A. The result (2.31) differs from the prediction $P \sim 1 - M^2/4$ of Moore & Pullin (1987) and the origin of this difference stems from the expression (2.21), which contains two scaling constants A and B. According to the present analysis, either one of these constants may be chosen at the outset, with the other to be determined; subsequent analysis leads to the equation (2.25) that links the two constants. The work of Moore & Pullin (1987) amounts to setting both A and B to be zero, and it is argued here that this disregards the requirement (2.25).

3. Outer approximation

For a wide class of wave problems, including the special case considered here, a local incompressible flow gives a good representation in the near field, which is the flow region that is well within a typical acoustic wavelength of the sound field from the source of the disturbance (but not too close to a vortex). An approximation for more distant points must take account of the wavelike nature of the fluid, but, at such distances, the source region has negligible size, and such considerations lead typically to a wave equation driven by appropriate multipole singularities that represent the effect of the local compact disturbance. This is formalized by introducing 'outer' coordinates (x_1, x_2), that are scaled with respect to a length h_0c_0/U_0 that characterizes a wavelength scale, where h_0 and U_0 are representative lengths and speeds for the problem.

Using this procedure in the present (steady-flow) problem we take $x_i' = (2h_0^2c_0/\kappa)x_i = (h_0/M)x_i$, and the relationship between main and outer variables is

$$x_1 = MX_1, \quad x_2 = MX_2, \quad r = MR, \tag{3.1}$$

and, when written in terms of the outer variable, expression (2.29) (together with (2.28)), is

$$\widehat{\Phi} = M\left\{-2\frac{\cos\theta}{r}\right\} + M^3\left\{(2A-1)\frac{\cos\theta}{r} + \frac{1}{2}\frac{\cos 3\theta}{r}\right\}, \tag{3.2}$$

and this has to be matched with an appropriate outer approximation. The scale of the outer potential, for the fluid motion relative to fluid at infinity, is shown from equation (3.2) to be proportional to M. Thus, we are led to seek an outer approximation of the form

$$\frac{\phi'}{\kappa} \equiv \Phi = -\left\{\tfrac{1}{2} + \left(\tfrac{1}{2}A + \tfrac{1}{8}\right)M^2\right\}X_1 + M\psi(x_1, x_2). \tag{3.3}$$

Substitution into the governing equation (2.7) leads to the outer equation

$$\nabla^2\psi = M^2\frac{\partial^2}{\partial x_1^2}\psi + o(M^2), \tag{3.4}$$

with a matching requirement, arising from the expression (3.2), that

$$\psi \sim \left\{-2\frac{\cos\theta}{r}\right\} + M^2\left\{(2A-1)\frac{\cos\theta}{r} + \frac{1}{2}\frac{\cos 3\theta}{r}\right\} \tag{3.5}$$

as $r \equiv (x_1^2 + x_2^2)^{1/2} \to 0$. This ensures that the inner limit of the outer solution matches with outer limit (3.2) of the main approximation (2.9).

In this particular case of steady flow (in a reference frame fixed with the vortices), the governing equation (3.4) can readily be reduced to Laplace's equation through the transformation $\xi_1 = \beta x_1$, $\xi_2 = x_2$, where

$$\beta = (1 - M^2)^{-1/2}. \tag{3.6}$$

The first term of expression (3.5) indicates a dipole singularity at the origin, with strength chosen to ensure matching with expresssion (3.5).

A line source of unit strength in a moving fluid (with velocity $(-MU, 0)$ and Mach number $2MU \sim M$ in our scaled outer coordinates) has potential

$$\psi_s = \frac{\beta}{2\pi}\ln\left\{\beta^2x_1^2 + x_2^2\right\}^{1/2}, \tag{3.7}$$

with β given by (3.6) so a dipole of strength $2\pi D(M)$ has potential $\psi_d = -2\pi D \partial\phi_s/\partial x_1$, thus,

$$\psi_d = -D(M)\beta^3 \frac{x_1}{\beta^2 x_1^2 + x_2^2}, \qquad (3.8)$$

with $D = D_0 + D_2 M^2 + \cdots$, say.

Expanding for small values of M (with (x_1, x_2) fixed) and writing the function in terms of polar coordinates (with $x_1 = r\cos\theta, x_2 = r\sin\theta$), ψ_d takes the form

$$\psi_d = -\frac{\cos\theta}{r}\left\{D_0 + M^2\left(D_2 + \tfrac{3}{4}D_0\right)\right\} + \frac{\cos 3\theta}{4r} D_0 M^2 + \cdots. \qquad (3.9)$$

Matching this with expression (3.5), we find that $D_0 = 2$ and $D_2 = -(2A + 1/2)$, thus

$$D \sim 2 - (2A + 1/2)M^2. \qquad (3.10)$$

Note in particular, from equations (3.10) and (2.27), that $D = 2h$, which is the separation (measured in main coordinates) between the two vortices, and the following physical argument shows that this is expected. The main solution is that of a self-propagating vortex pair at $(x_1, x_2) = (MUt, \pm Mh)$ in outer coordinates fixed in the fluid, with $h = (1 - \delta M^2)$. Now the incompressible flow field due to a vortex pair of strengths $\pm K$ is exactly equivalent to that due to a distribution of line dipoles of density $K = 2\pi\kappa$ perpendicular to the line joining the two vortices. Thus, it is natural to represent the outer field as such a dipole distribution in compressible fluid. Further, the line joining the vortices has length $2Mh$ (measured in outer coordinates) and this compact distribution can be expressed as a single line dipole at the mid-point between the vortex pair and of total strength $4\pi\kappa Mh$. Noting the two scaling constants in expression (3.3), this implies that ψ corresponds to a dipole of strength $4\pi h = 2\pi D$ with $D = 2h$ as above.

4. Core region approximation

It has been pointed out by Barsony-Nagy *et al.* (1987), and also by Moore & Pullin (1987), that the main approximation (2.9) also fails at distances $\hat{R} = O(M)$ from the vortex. An indication of this comes from the form of the approximation (2.24), in which the two leading terms for $\Phi - \hat{\theta}$, with $\hat{R}$ small, are proportional to $\sin 2\hat{\theta}$ and have the same magnitude when $\hat{R} = O(M)$. This suggests the structure of an 'inner' region with

$$\Phi \equiv \kappa^{-1}\phi' \sim \hat{\theta} + M^2\phi_{in}(s, \hat{\theta}) \quad \text{with} \quad \hat{R} = Ms. \qquad (4.1)$$

Substitution in the governing equation (2.7) leads to a partial differential equation for ϕ_{in}, due to Taylor (1930) (see also Barsony-Nagy *et al.* 1987 or Moore & Pullin 1987). Thus,

$$\left\{1 - \frac{2(\gamma-1)}{s^2}\right\}\frac{\partial^2\phi_{in}}{\partial s^2} + \left\{1 - \frac{2(\gamma-3)}{s^2}\right\}\frac{1}{s}\frac{\partial\phi_{in}}{\partial s} + \left\{1 - \frac{2(\gamma+1)}{s^2}\right\}\frac{1}{s^2}\frac{\partial^2\phi_{in}}{\partial\hat{\theta}^2} = 0. \qquad (4.2)$$

Barsony-Nagy *et al.* (1987) point out that the coefficient of the term $\partial^2\phi_{in}/\partial s^2$ vanishes on the circle $s = (2(\gamma-1))^{1/2}$, (that is, at $\hat{R} = 2^{1/2}M(\gamma-1)^{1/2}$), and that the potential solution has no physical meaning inside this circle; the model of a 'line vortex' is inadequate at such small distances and we require a more refined model.

With the particular $\hat{\theta}$-dependence shown in the leading terms of equation (2.26), when $\hat{R} = O(M)$, we seek a solution of the form $\phi_{in} = \chi \sin 2\hat{\theta}$, leading to an ordinary differential for χ that is shown by Barsony-Nagy *et al.* (1987) to be related to the hypergeometric equation in terms of the independent variable

$$\tau = 2(\gamma - 1)s^{-2}. \tag{4.3}$$

We find that the functions $\tau\chi$ and $\tau^{-1}\chi$ satisfy hypergeometric equations. Thus, we can express the general solution for χ in the form

$$\chi(\tau) = A_1\,\tau^{-1}\,{}_2F_1(a, b; c; 1-\tau) + B_1\tau\,{}_2F_1(a+2, b+2; 3; \tau), \tag{4.4}$$

with parameters a, b, c given by

$$a = \frac{(3-2\gamma) + (4\gamma^2-3)^{1/2}}{2(\gamma-1)}, \quad b = \frac{(3-2\gamma) - (4\gamma^2-3)^{1/2}}{2(\gamma-1)}, \quad c = \frac{1}{\gamma-1}. \tag{4.5}$$

Note that the first term in (4.4) is analytic at $\tau = 1$ (that is, at $s^2 = 2(\gamma-1)$), but not at $\tau = 0$ ($s \to \infty$) and that the second function is analytic at $\tau = 0$, but not at $\tau = 1$.

The coefficient A_1 in (4.4) is determined by matching, as $s \to \infty$ ($\tau \to 0$) with the main solution (2.26), and this requires

$$\chi \to -\frac{(\gamma-1)}{4\tau} - \tfrac{3}{4} \quad \text{as} \quad \tau \to 0. \tag{4.6}$$

The second hypergeometric function in expression (4.4) is regular at $\tau = 0$ and the first function has behaviour given by Abramowitz & Stegun (1965, formula 15.3.11, p. 559), whence

$$\chi \sim A_1 \frac{\Gamma(c)}{\Gamma(a+2)\Gamma(b+2)} \left\{ \frac{1}{\tau} + \frac{3}{\gamma-1} + O(\tau \ln \tau) \right\} + B_1\{\tau + O(\tau^2)\}. \tag{4.7}$$

Matching the τ^{-1} terms in expressions (4.6) and (4.7) requires

$$A_1 = -\frac{(\gamma-1)\Gamma(a+2)\Gamma(b+2)}{4\Gamma(c)}. \tag{4.8}$$

The fact that this value for A_1 also ensures the matching of the $O(1)$ terms in expressions (4.6) and (4.7) is a good consistency check on the approximation scheme.

The constant B_1, of (4.4), is not determined at this stage. It was remarked earlier that the model of a line-vortex with zero diameter breaks down as the distance from the vortex approaches zero and we require a more refined model, such as that of a thin core containing stagnant constant-pressure fluid (Moore & Pullin 1987), or that where there is a light cylinder of small radius that drifts along with the fluid. The value of the constant B_1, in (4.4), depends on the details of the model problem in the region close to the vortex centre.

Barsony-Nagy *et al.* (1987) take $B_1 = 0$ to ensure that the solution is bounded at $\tau = 1$ (that is, at $s^2 = 2(\gamma-1)$).

5. Conclusions

A calculation has been made to determine the flow field induced by a pair of self-propagating line vortices, with equal and opposite circulations, in a compressible fluid.

Matched expansions have been used, with respect to small values of the Mach number M. The 'main' approximation is taken for those points in the flow field that are neither too far from, nor close to, the vortex lines. In this main region, the elementary textbook solution for incompressible flow is enhanced by a term of order $O(M^2)$, according to the Rayleigh–Janzen approximation.

This approximation fails at points that are either far from, or very close to, a vortex. At distant points, the leading-order solution is expressed in terms of the potential due to a moving dipole in compressible fluid, with strength chosen to ensure matching; this approximation has been bolstered by an alternative physical argument that is used to reach the same conclusion.

The main approximation also fails at points close to either vortex line. The governing equations in these regions (one near each vortex) have been described by Taylor (1930) and Barsony-Nagy *et al.* (1987), and their analysis has been incorporated in the present problem. The line vortex model ultimately breaks down when the distance from a vortex becomes sufficiently small, because of the non-physical nature of the vortex tube of zero thickness. In such a small region, we must amend the model to take account of a core of finite, rather than zero, diameter.

A parameter of particular interest is the 'propagation number' $P = 4\pi U'h'/K = P(M)$, which depends on the Mach number M. The conclusion of this work is that $P = 1 + o(M^2)$. That is, there is no change in $P(M)$ to order $O(M^2)$ and this differs from the prediction $P \sim 1 - M^2/4$ in earlier work.

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